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The Computation of Second Order Energies Using a Reference Spectrum Euler Function

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We compute the Euler Function which appears in the evaluation of second order energies of Fermi Liquids for the case where the single particle energies are given by a reference spectrum approximation, namely $\epsilon_p(\mathbf{k}) = \hbar^2\mathbf{k}^2/2m_p^* + \Delta_p$ (with a similar expression for hole energies). It is found that the Euler Function for the case $m_p^* = m_h^*$ but $\Delta_p \neq \Delta_h$ can be obtained analytically, and is a useful approximation to the exact Euler Function.

I. INTRODUCTION

In many Fermi liquid calculations one requires the second order contribution

$$E_{U,V} = \sum_{\mathbf{k}_1\mathbf{k}_2}{}' (k_1k_2 \mid U\frac{Q}{e}V \mid k_1k_2\rangle \tag{1}$$

where the primed summation runs over $|\mathbf{k}_1|, |\mathbf{k}_2| < k_F$, U and V are two body operators (usually potentials), Q is the Pauli exclusion operator and e is the energy denominator. As examples we cite the long range potential contribution to the energy of nuclear matter [1]

$$E_2 = \sum_{\mathbf{k}_1\mathbf{k}_2}{}' \langle \mathbf{k}_1\mathbf{k}_2 \mid v_\ell \frac{Q}{e} v_\ell \mid \mathbf{k}_1\mathbf{k}_2\rangle, \tag{2}$$

and the contribution of three body forces and long range forces to the nuclear matter energy [2].

$$E_{23} = \sum_{\mathbf{k}_1\mathbf{k}_2}{}' \langle \mathbf{k}_1\mathbf{k}_2 \mid v_\ell \frac{Q}{e} v_3 \mid \mathbf{k}_1\mathbf{k}_2\rangle. \tag{3}$$

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To evaluate expressions of the type (1) we introduce a complete set of intermediate states $\mathbf{m}_1$, $\mathbf{m}_2$, so that (1) becomes more explicitly

$$E_{UV} = \sum_{\substack{|\mathbf{k}_1|,|\mathbf{k}_2|<\mathbf{kk} \\ |\mathbf{m}_1|,|\mathbf{m}_2|>\mathbf{kk}}} \frac{\langle \mathbf{k}_1\mathbf{k}_2 \mid U \mid \mathbf{m}_1\mathbf{m}_2\rangle\langle \mathbf{m}_1\mathbf{m}_2 \mid V \mid \mathbf{k}_1\mathbf{k}_2\rangle}{\epsilon(\mathbf{k}_1) + \epsilon(\mathbf{k}_2) - \epsilon(\mathbf{m}_1) - \epsilon(\mathbf{m}_2)}. \tag{4}$$

The energy denominator e has now been explicitly written in terms of the single particle energies $\epsilon(\mathbf{k})$. To further evaluate (4) we make the assumption that the matrix elements are of a local potential and conserve the total momentum,[1] writing

$$\mathbf{m}_1 = \mathbf{k}_1 + \mathbf{q}$$

$$\mathbf{m}_2 = \mathbf{k}_2 - \mathbf{q}$$

$\langle \mathbf{k}_1\mathbf{k}_2 \mid U \mid \mathbf{m}_1\mathbf{m}_2\rangle = (1/\Omega)\,\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{m}_1+\mathbf{m}_2}\,U(\mathbf{q})$, where Ω is the normalization volume, with a similar definition of $V(\mathbf{q})$, and we replace the summations by integrations in the usual way.

Then we rewrite (4) as

$$\frac{E_{UV}}{N} = \frac{3}{2}\frac{1}{(2\pi)^5}\frac{k_F m}{\hbar^2}\sum_{\text{spins}}\int d^3\mathbf{q}\; U(\mathbf{q})\,V(-\mathbf{q})\,g_\epsilon\left(\frac{\mathbf{q}}{k_F}\right) \tag{5}$$

where

$$g_\epsilon(\mathbf{X}) = -\frac{\hbar^2}{2m}\frac{1}{8\pi^2 k_F{}^4}\int_D d\mathbf{k}_1\, d\mathbf{k}_2 \frac{1}{\epsilon(\mathbf{k}_1) + \epsilon(\mathbf{k}_2) - \epsilon(\mathbf{k}_1 + k_F\mathbf{X}) - \epsilon(\mathbf{k}_2 - k_F\mathbf{X})} \tag{6}$$

which is a function only of $|\,\mathbf{X}\,|$, but is a functional of ϵ. The domain of integration D is determined by the requirements that $\mathbf{k}_1$ and $\mathbf{k}_2$ are inside the Fermi Sea, while $\mathbf{k}_1 + \mathbf{X}k_F$ and $\mathbf{k}_2 - \mathbf{X}k_F$ are outside.

For the case of a free energy spectrum,

$$\epsilon(\mathbf{k}) = \frac{\hbar^2 k^2}{2m}, \tag{7}$$

this function g was evaluated by Euler [3]. In this case we set $g_\epsilon = g_F$

$$g_F(|\,\mathbf{X}\,|) = \frac{1}{16\pi^2}\int_{D'} d\boldsymbol{\varkappa}_1\, d\boldsymbol{\varkappa}_2 \frac{1}{\mathbf{X}\cdot(\mathbf{X} + \boldsymbol{\varkappa}_1 - \boldsymbol{\varkappa}_2)} \tag{8}$$

where dimensionless momenta $\boldsymbol{\varkappa}_i = \mathbf{k}_i/k_F$ have been introduced and the domain D' is

$$D' = \{\boldsymbol{\varkappa}_1, \boldsymbol{\varkappa}_2 \mid |\,\boldsymbol{\varkappa}_1\,| < 1, |\,\boldsymbol{\varkappa}_2\,| < 1, |\,\boldsymbol{\varkappa}_1 + \mathbf{X}\,| > 1, |\,\boldsymbol{\varkappa}_2 - \mathbf{X}\,| > 1\}. \tag{9}$$

[1] This is the case in an infinite system, such as nuclear matter or liquid ^{3}He. It would not be the case for a shell model calculation for a finite nucleus in an harmonic oscillator basis.

Following Levinger *et al.* [4], we introduce the Euler Function P_ϵ so that with $2u = |\mathbf{X}|$

$$uP_\epsilon(u) = 120u^2 g_\epsilon(2u) \tag{10}$$

Brown [5] also introduces an Euler Function P which is 1/15th of that of this work.

Equation (5) may then be rewritten as

$$\frac{E_{UV}}{N} = \frac{45}{2}\frac{1}{(2\pi)^5}\frac{k_F{}^4 m}{\hbar^2}\sum_{\text{spins}}\int d\Omega_q \int_0^\infty uP_\epsilon(u)\, du\, U(2k_F u, \Omega_q)\, V(2k_F u, \Omega_{-q}) \tag{11}$$

writing the vector $\mathbf{q}$ in terms of its magnitude $q = 2uk_F$, and its direction Ω_q. In the free spectrum case, Euler obtained the result

$$\begin{aligned} P_F(u) &= P_1(u) \qquad |u| < 1 \\ &= P_2(u) \qquad |u| > 1 \end{aligned} \tag{12}$$

where

$$\begin{aligned} P_1(u) = {} & (4 + \tfrac{15}{2}u - 5u^3 + \tfrac{3}{2}u^5)\ln(u+1) \\ & + (29u^2 - 3u^4) - 40u^2 \ln 2 \\ & + (4 - \tfrac{15}{2}u + 5u^3 - \tfrac{3}{2}u^5)\ln(1-u) \end{aligned} \tag{13}$$

$$\begin{aligned} P_2(u) = {} & (4 - 20u^2 - 20u^3 + 4u^5)\ln(u+1) + 4u^3 \\ & + 22u + (-4 + 20u^2 - 20u^3 + 4u^5)\ln(u-1) \\ & + (40u^3 - 8u^5)\ln u. \end{aligned}$$

The subscript F distinguishes the Free Spectrum Euler Function P_F from the general case P_ϵ. This function is discussed in more detail by Levinger *et al.* [4], da Providencia [6] and in the appendix.

At present the most commonly used approximation to $\epsilon(\mathbf{k})$ is the quadratic Reference Spectrum [7]

$$\epsilon(\mathbf{k}) = \frac{\hbar^2}{2m_h^*}k^2 + \Delta_h \qquad \text{for} \quad |\mathbf{k}| < k_F \tag{15a}$$

$$\epsilon(\mathbf{k}) = \frac{\hbar^2}{2m_p^*}k^2 + \Delta_p \qquad \text{for} \quad |\mathbf{k}| > k_F. \tag{15b}$$

The subscripts h and p refer to holes and particles. There is currently some controversy over the best values of the parameters [8]. But a common choice is [9]

$$\begin{aligned} m_h^* &= 0.65m \\ \Delta_h &= 2.12\,\frac{\hbar^2 k_F{}^2}{2m} \\ m_p^* &= m \\ \Delta_p &= 0. \end{aligned} \tag{16}$$

It is our purpose in this paper to discuss the Generalized Euler Function for this case. In Section 2 we consider the special case $m_h^* = m_p^*$. In this case we can obtain a useful analytic expression for $g(\mathbf{q})$. Then in Section 3 the case $m_h^* \neq m_p^*$ is considered. An analytic result is not possible in this case, but we can reduce the problem to a double numerical integration.[2] It turns out that the analytic expression of section (2) is a good approximation to $g(\mathbf{q})$ in this case also, at least for the parameter values of interest. Section 4 contains an example of the use of this generalized Euler Function in the computation of second order energies.

2. $m_h^* = m_p^* = m^*$

This case is similar to that considered by Euler, in that no quadratic terms in $\mathbf{k}_1$ or $\mathbf{k}_2$ appear. It is this which makes the analytic result possible. In this case

$$g_{m_h^*=m_p^*}(\mathbf{X}) = \frac{m^*}{m} \int_{D'} \frac{d\boldsymbol{\varkappa}_1 \, d\boldsymbol{\varkappa}_2}{16\pi^2} \frac{1}{\delta + \mathbf{X} \cdot (\mathbf{X} + \mathbf{k}_1 - \mathbf{k}_2)} \tag{17}$$

where

$$\delta = \frac{2m_p^*}{k_F{}^2\hbar^2} (\Delta_h - \Delta_p). \tag{18}$$

The appropriate Euler Function we write as $P(u, m^*, \delta)$

$$\begin{aligned} g_{m_h^*=m_p^*}(q) &= (120u)^{-1} P(u, m^*, \delta) \\ P(u, m^*, \delta) &= \frac{m^*}{m} \int_{D'} \frac{d\boldsymbol{\varkappa}_1 \, d\boldsymbol{\varkappa}_2}{(2\pi)^2} \frac{30u}{\delta + 2u(2u + \kappa_{1x} - \kappa_{2x})} \end{aligned} \tag{19}$$

where, we can take $\mathbf{X}$ to lie along the x axis without loss of generality, since the integral depends only on $| \mathbf{X} | = 2u$.

Once again the integral takes on two forms according as $u > 1$ or $u < 1$. For $u > 1$, the domain of integration D' becomes independent of u, since $| \boldsymbol{\varkappa} \pm \mathbf{X} | > 1$ for $| \mathbf{X} | > 2$ or $u > 1$. In this case u appears only in the integrand, and comparing the integral with that in the appendix we see that

$$P(u, m^*, \delta) = \frac{m^*}{m} P_2\left(u + \frac{\delta}{2u}\right) \qquad \text{for} \quad u > 1. \tag{20}$$

This relation was noticed by Brown *et al.* [10], and was proposed by them as an approximation to $P(u, m^*, \delta)$ for all u in the spirit of the reference spectrum approximation for calculating nuclear matter energies, in which the Pauli Principle is ignored [7]. The result (20) was also given in detail by Sprung [11].

[2] It is possible to do a further integration analytically, reducing the evaluation to just one numerical integration. But the numerical evaluation in this case is quite unstable, and the double numerical integration is to be preferred.

Equation (20) implies that as $u \to \infty$, $P(u, m^*, \delta) \to m^*/m \cdot 10/3 \cdot u^{-1}$. However, in this case an analytic result can be obtained also for $u < 1$. The algebra is somewhat tedious, and is outlined in the appendix. The final result is, for $u < 1$,

$$P(u, m^*, \delta) = \frac{m^*}{m} 120u \sum_{i=0}^{6} J_i(u, \delta) \tag{21}$$

$$J_0 = \frac{u}{24}\left(\frac{3}{2}\frac{\delta}{u} + 5 - u^2\left[\frac{\delta}{2u} + 1\right]\right) \tag{22}$$

and for $i = 1,..., 6$

$$J_i = [y^2\{\phi_i(y) \ln 2y - \psi_i(y)\}]_{y=y_L^{(i)}}^{y=y_U^{(i)}}. \tag{23}$$

TABLE I

The Polynomials $\phi_i(y)$, $\psi_i(y)$[a] and Limits $y_L^{(i)}$ and $y_U^{(i)}$ [b]

i	$a_0^{(i)}$	$a_1^{(i)}$	$a_2^{(i)}$	$a_3^{(i)}$	$y_L^{(i)}$	$y_U^{(i)}$
1	$\frac{1}{4} - \frac{1}{4}(2e_1 + u)^2$	$\frac{2}{3}(2e_1 + u)$	$-\frac{1}{2}$	0	e_3	e_5
2	$-\delta/2$	$\frac{4}{3}u$	0	0	e_1	e_3
3	$e_1e_2/2u$	$\frac{-e_1^2 + 3e_1 + 1}{3u}$	$e_2/2u$	$-\frac{1}{5u}$	e_4	e_2
4	e_3	$\frac{1}{3}(u - 2e_1 - 3)$	$\frac{1}{2}$	0	e_3	e_4
5	$(e_1 + a)e_6/2u$	$\frac{-(e_1 + u)^2 + 3(e_1 + u) + 1}{3u}$	$e_6/2u$	$-\frac{1}{5u}$	e_6	e_2
6	e_5	$\frac{1}{3}(-u - 2e_1 - 3)$	$\frac{1}{2}$	0	e_2	e_5

[a] $\phi_i(y) = \sum_{j=0}^{3} a_j^{(i)} y^j$, $\psi_i(y) = \sum_{j=0}^{3} (j + 2)^{-1} a_j^{(i)} y^j$, and $a_j^{(i)}$ are tabulated.

[b] $y_L^{(i)}$ and $y_U^{(i)}$ are selected from the six values $e_1 = \delta/4u$, $e_2 = e_1 + 1$, $e_3 = e_1 + \frac{1}{2} - \frac{1}{2}u$, $e_4 = e_1 + 1 - u$, $e_5 = e_1 + \frac{1}{2} + \frac{1}{2}u$, $e_6 = e_1 + 1 + u$.

The values of $y_L^{(i)}$, $y_U^{(i)}$, and the polynomials ϕ_i and ψ_i are given in Table I. From this table it is apparent that the limit $u \to 0$ is somewhat delicate in the expansion of Eq. (21). It is most easily obtained from the integral (19), since as $u \to 0$ the integrand becomes $2u/\delta$, approximately independent of κ_1 and κ_2. The integrals are then independent and straightforward (see e.g. Pines [12]), and give

$$P(u, m^*, \delta) \approx \frac{m^*}{m} \frac{30u^3}{\delta} \qquad \text{as} \quad u \to 0 \text{ for finite } \delta. \tag{24}$$

This is to be compared with

$$P_1(u) \approx \frac{120}{3}(1 - \ln 2)\, u^2 \qquad \text{as} \quad u \to 0, \tag{25}$$

showing that the introduction of the energy gap Δ causes a more rapid approach to zero.

Figure 1 shows $P(u, m^*, \delta)$ for $m^* = m^* = 0.65m$ and $\delta = 0$ and $\delta = .65 \times 2.12 = 1.38$. $P_F(u)$ is shown for comparison. As would be expected, an effective mass less than the free nucleon mass and the energy shift Δ both reduce P, since they increase the energy denominator.

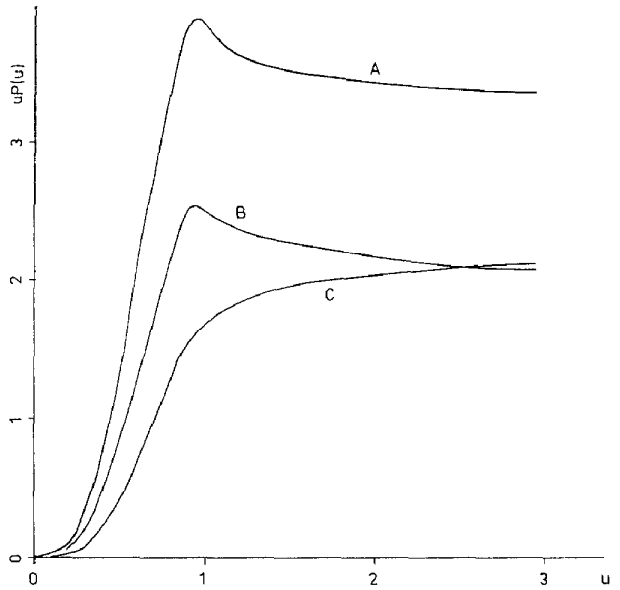

FIG. 1. Curve A is $uP_F(u)$, curve B is $(m^*/m)\, u\, P_F(u)$, and curve C is $uP(u, m^*, \delta)$. The numerical values used are $m^* = 0.65\, m$ and $\delta = 0.65 \times 2.12$.

3. THE GENERAL CASE

When the effective mass of the holes is different from the effective mass of the particles it is no longer possible to do all of the integrations analytically. In this case the energy denominator becomes

$$e = -\frac{\hbar^2 k_F^2}{m_p^*}\left\{\delta + \mathbf{X}\cdot(\mathbf{X} + \boldsymbol{\varkappa}_1 - \boldsymbol{\varkappa}_2) + \frac{1}{2}\left(1 - \frac{m_p^*}{m_h^*}\right)(\kappa_1^2 + \kappa_2^2)\right\}. \tag{26}$$

Following the same type of manipulations as before, we can carry out the y, z, y', z'

integrals analytically, but the quadratic terms in Eq. (26) prevent further progress, and the x and x' integrals must be evaluated numerically.[2] The result is

$$P_{RS}(u) = -30\,\frac{m_p^*}{m}\,u\left(\frac{m_p^*}{m_h^*}-1\right)^{-1}\int_E dx\,dx'\,I(x,x',u) \tag{27}$$

where we have used the subscript RS for the general Reference Spectrum Euler Function. The integral is over the domain $E(u) = E_1(u) \cup E_2(u) \cup E_3(u)$, where

$$E_1(u) = \{(x,x') \mid -1 < x < d(u),\ -1 < x' < x\} \tag{28a}$$

$$E_2(u) = \{(x,x') \mid -1 < x < d(u),\ d(u) < x' < u\} \quad \text{if} \quad 0 < u < 1$$
$$= \varnothing \quad \text{if} \quad u > 1. \tag{28b}$$

$$E_3(u) = \{(x,x') \mid d(u) < x < u,\ d(u) < x' < x\} \quad \text{if} \quad 0 < u < 1$$
$$= \varnothing \quad \text{if} \quad u > 1 \tag{28c}$$

and

$$d(u) = \min\{-1 + 2u,\ 1\}. \tag{28d}$$

The integrand $I(x, x', u)$ is given by

$$I(x,x',u) = \rho \ln\frac{\rho(\rho-\sigma-\tau)}{(\rho-\sigma)(\rho-\tau)} + \sigma\ln\frac{\rho-\sigma}{\rho-\sigma-\tau} + \tau\ln\frac{\rho-\tau}{\rho-\sigma-\tau} \tag{29}$$

ρ, σ, and τ are functions of x, x', and u.

$$\rho = 2 - 4\left(\frac{m_p^*}{m_h^*}-1\right)^{-1} u\left(\frac{2m_p^*\delta}{u} + 2u - x - x'\right) \tag{30}$$

and the functional expressions for σ and τ depend on the location of (x, x') in the domain of integration.

$$\text{For} \quad (x,x') \in E_1, \qquad \sigma = 1 - x^2 \qquad \text{and} \qquad \tau = x'^2, \tag{31a}$$

$$\text{for} \quad (x,x') \in E_2, \qquad \sigma = 1 - x^2 \qquad \text{and} \qquad \tau = 4u(u - x'), \tag{31b}$$

and

$$\text{for} \quad (x,x') \in E_3, \qquad \sigma = 4u(u-x) \qquad \text{and} \qquad \tau = 4u(u-x'). \tag{31c}$$

The limits $u \to 0$ and $u \to \infty$ can again be calculated independently. As $u \to \infty$ the terms involving **X** in Eq. (26) dominate the energy denominator, so $P_{RS}(u)$ approaches the free Euler Function, up to a factor m_p^*/m.

$$P_{RS}(u) \approx \frac{m_p^*}{m}\cdot\frac{10}{3}\,u^{-1} \qquad \text{as} \quad u \to \infty. \tag{32}$$

For small values of u or $|\mathbf{X}|$ we can ignore the terms in **X** in the energy denominator.

Moreover, for small **X**, $\varkappa_1$ and $\varkappa_2$ must lie very close to the Fermi surface to be in the domain of integration D', so we can set

$$\kappa_1^2 = \kappa_2^2 = 1 + O(u) \tag{33}$$

in Eq. (26) in this limit. The integrals can now be done as in the preceding section, to give

$$P_{RS}(u) \approx \frac{m_p^*}{m} \frac{30u^3}{\delta - (m_p^*/m_h^* - 1)}. \tag{34}$$

In Figure 2 the functions $uP(u, m^*, \delta)$ and $uP_{RS}(u)$ are compared, for $\delta = 2.12$ and $m^* = m_p^* = 1$, $m_h^* = 0.65$. It will be seen that the differences are slight. Also plotted is $uP(u, m^* = 0.65, \delta = 0.65 \times 2.12)$ which is seen to be significantly smaller. This is not surprising since m^*/m essentially sets the scale. It is clear that the best approximation to P_{RS} is obtained using $m_p^* = m^*$ in $P(u, m^*, \delta)$.

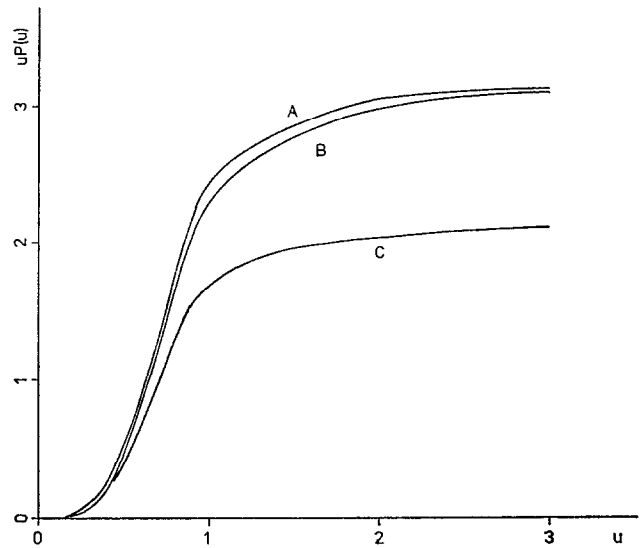

FIG. 2. Curve A is $uP_{RS}(u)$ for $m_p^* = m$, $m_h^* = 0.65$, $\delta = 2.12$. Curve B is $uP(u, m^* = 1, \delta = 2.12)$. Curve C is $uP(u, m^* = 0.65, \delta = 0.65 \times 2.12)$.

4. EXAMPLE AND CONCLUSIONS

As an example of the use of the Generalised Euler Function we give the calculation of the second order energy of Eq. (3) which contains the long range two body potential and an effective 2 body potential constructed from three body forces. This quantity was evaluated in our earlier work [2] for the case $P_\epsilon = P_F$, i.e., with a free particle energy

spectrum.[3] The necessity of evaluating it for the reference spectrum led to the developments described above.

We refer the reader to Ref. [2] for a detailed description of the potentials involved. The work here uses the same 3 body potential as Ref. [2] although that is not currently regarded as the best choice [12]. The results of the calculation for the choices $P_\epsilon(u) = P_F(u)$, $P_\epsilon(u) = P(u, m^* = 0.65, \delta = 0.65 \times 2.52)$, $P_\epsilon(u) = P(u, m^* = 1, \delta = 2.12)$, $P_\epsilon(u) = P_{RS}(u, m_p^* = 1, m_h^* = 0.65, \delta = 2.12)$ are shown in Table II.

TABLE II

The Second Order Energy $\langle V_l\, Q/e\, V_3 \rangle$ for Various Energy Spectra

	Form factor II	Form factor III
$P_F(u)$	−4.79	−5.15
$P(u, m^* = 0.65; \delta = 0.65 \times 2.12)$	−2.00	−2.06
$P(u, m^* = 1, \delta = 2.12)$	−2.62	−2.66
$P_{RS}(u, m_p^* = 1, m_h^* = 0.65, \delta = 2.12)$	−2.80	−2.86

As one would have expected from an examination of Figures 1 and 2, introducing an energy gap in the spectrum substantially reduces the second order contribution to the energy. Moreover it is clear that the approximation of setting m_p^* equal to m_h^* is good to better than 10% in computing second order energies.

A number of other generalised Euler Functions have been developed in the literature These have concentrated, not on the direct term when $m_p^* \neq m_h^*$ which is the subject of this paper, but on generalisations to exchange terms [11, 14] and non local potentials [15]. In these cases it was also found necessary to perform additional integrations numerically. The Euler method has also been extended to handle third order terms [16], additional numerical integrations being again required.

To summarise, we have obtained an analytic expression for the reference spectrum Euler Function for the case $m_p^* = m_h^*$, and have shown that it is a good approximation to the exact reference spectrum Euler Function, at least for $m_p^* > 0.65 m_h^*$.

Appendix

1. *Euler's Function* $P_F(u)$

To evaluate the integral for $P_F(u)$ given in Eq. 8, it is most convenient to write the integral in cartesian coordinates, taking $\mathbf{X}$ in the x direction, and writing $\kappa_1 = (x, y, z)$ and $\kappa_2 = (x', y', z')$.

$$P_F(u) = \frac{120u}{16\pi^2} \int_{D'} d\kappa_1\, d\kappa_2 \frac{1}{2u(2u + x - x')} .$$

[3] Unfortunately the effective potential in this earlier work was slightly in error. This has been corrected here, so the figure in table II is not identical to the result published there.

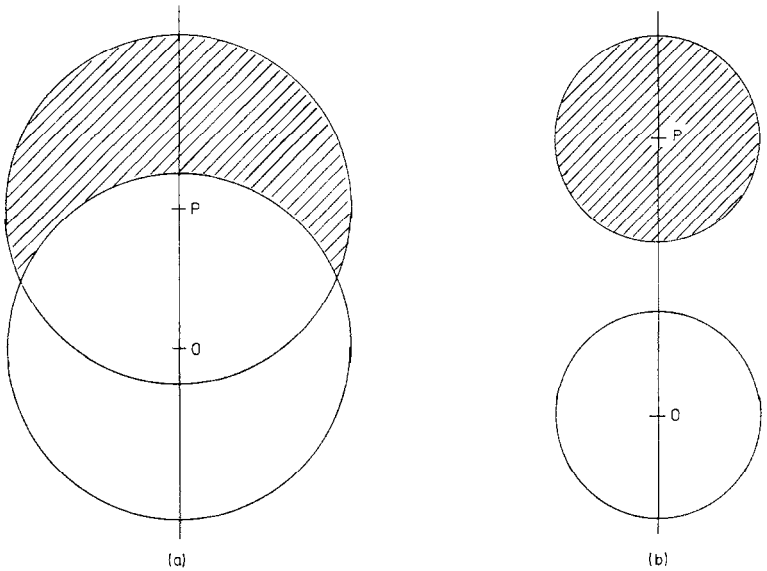

FIG. 3. The circles are unit circles centred on 0 and P, which are separated by $2u$. The shaded region, rotated about $0P$, is the region of integration for κ_1. That for κ_2 is similar. In figure (a) $0 < u < 1$, and in figure (b) $u > 1$.

The domain of integration D' is shown in Fig. 3. Two cases can be clearly distinguished, according as $u > 1$ or $u < 1$. Accordingly we set

$$P_F(u) = P_1(u) \qquad \text{for} \quad 0 < u < 1$$

and

$$P_F(u) = P_2(u) \qquad \text{for} \quad u > 1.$$

We consider the case $u > 1$ first, because it is the simpler, in that

$$P(u) = \frac{120u}{16\pi^2} \int_{\substack{x^2+y^2+z^2<1 \\ x'^2+y'^2+z'^2<1}} dx\, dy\, dz\, dx'\, dy'\, dz' \, \frac{1}{2u(2u + x - x')}$$

and the domain of integration is particularly simple. The integrals over y, z and y', z' introduce factors $2\pi(1 - x^2)$ and $2\pi(1 - x'^2)$ respectively so that

$$P_2(u) = \frac{15}{2} \int_{-1}^{1} dx(1 - x^2) \int_{-1}^{1} dx'(1 - x'^2) \frac{1}{2u + x - x'} .$$

Finally the x' and then the x integral can be performed to give the result of Eq. (14).

When $|u| < 1$, the domain D' depends on u, as is shown in Fig. 3(a). The integrals over y, z and y', z' can still be readily performed to give

$$P_1(u) = \frac{15}{2} \int_G dx\, dx' \, \frac{\kappa(x, x')}{2u + x - x'} .$$

It is convenient to write $G = G_1 \cup G_2 \cup G_3 \cup G_4$ where

$$G_1 = \{(x, x') \mid u < x < 1, u < x' < 1\}$$
$$G_2 = \{(x, x') \mid u < x < 1, 1 < x' < 1 + 2u\}$$
$$G_3 = \{(x, x') \mid 1 < x < 1 + 2u, u < x' < 1\}$$
$$G_4 = \{(x, x') \mid 1 < x < 1 + 2u, 1 < x' < 1 + 2u\}.$$

The function κ depends on the location of (x, x'),

$$\kappa(x, x') = 16u^2(u - x)(u - x') \quad \text{for} \quad (x, x') \in G_1$$
$$\kappa(x, x') = 4u(u - x)(1 - x'^2) \quad \text{for} \quad (x, x') \in G_2$$
$$\kappa(x, x') = 4u(1 - x^2)(u - x') \quad \text{for} \quad (x, x') \in G_3$$
$$\kappa(x, x') = (1 - x^2)(1 - x'^2) \quad \text{for} \quad (x, x') \in G_4 .$$

Once again the integral over (x, x') can be done analytically to obtain Eq. (13).

Euler gave useful approximations valid as $u \to 0$ and as $u \to \infty$.

2. *The Case* $m_h^* = m_p^*$

In this case the integrand is different, but the y, z, y', z' integrals can still be performed as described above. Then we have the integral, for $| u | < 1$

$$P(u, m^*, \delta) = \frac{15}{2} \frac{m^*}{m} \int_G dx\, dx' \frac{\kappa(x, x')}{\delta/2u + 2u + x - x'} .$$

The integrals are still tractable, since the denominator is a linear form in x and x', which gives logarithmic terms at worst, on performing the x' integral. One then has to integrate terms like $\int Q(x) \ln x\, dx$ where $Q(x)$ is a polynomial. With $R(x) = \int Q(x)\, dx$ this integral may be written as $\int R'(x) \ln x\, dx$, which yields to an integration by parts.

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